Atmospheric and Oceanic Fluid Dynamics

Fundamentals and Large-Scale Circulation

Geoffrey K. Vallis

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ATMOSPHERIC AND OCEANIC FLUID DYNAMICS

Fundamentals and Large-scale Circulation

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An asterisk indicates more advanced material that may be omitted on a first reading. A dagger indicates material that is still a topic of research or that is not settled.

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